Q.1 Find the domain of the following function.

(a) $f(x) = \dfrac{e^{4x-3}}{\sqrt{4x-7}}$

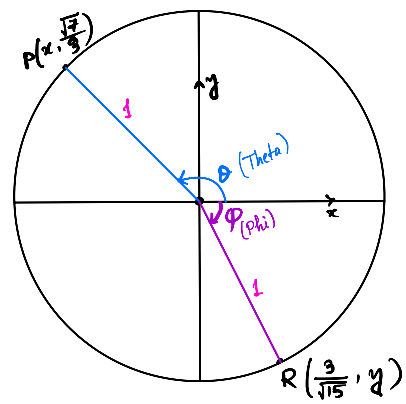
(b) $g(x) = \dfrac{\sqrt{x^2+1}}{\sin(3x+7)}$

(a) $f(x) = \dfrac{e^{4x-3}}{\sqrt{4x-7}}$

$$\text{Domain}(f) = \{x \text{ is real} \mid \sqrt{4x-7} \neq 0\}$$
$$= \{x \text{ is real} \mid 4x-7 \neq 0\}$$
$$= \{x \text{ is real} \mid 4x \neq 7\}$$
$$= \{x \text{ is real} \mid x \neq \tfrac{7}{4}\}$$

(b) $$\text{Domain}(g) = \{x \text{ is real} \mid \sin(3x+7) \neq 0\}$$
$$= \{x \text{ is real} \mid 3x+7 \neq n\pi, \text{ where } n \text{ is integer}\}$$
$$= \{x \text{ is real} \mid 3x \neq n\pi - 7, \text{ where } n \text{ is integer}\}$$
$$= \{x \text{ is real} \mid x \neq \frac{n\pi-7}{3}, \text{ where } n \text{ is any integer}\}$$

Q.2



Find $\tan\theta$ & $\cot\varphi$.

Since both P & Q are on a unit circle,

$$x^2 + \left(\frac{\sqrt{7}}{3}\right)^2 = 1^2$$

$$\Rightarrow x^2 + \frac{(\sqrt{7})^2}{3^2} = 1$$

$$\Rightarrow x^2 + \frac{7}{9} = 1$$

$$\Rightarrow x^2 = 1 - \frac{7}{9} = \frac{2}{9} = \left(\sqrt{\frac{2}{9}}\right)^2 = \left(\frac{\sqrt{2}}{3}\right)^2$$

$$\Rightarrow x = \pm \frac{\sqrt{2}}{3}$$

$$\left(\frac{3}{\sqrt{15}}\right)^2 + y^2 = 1^2$$

$$\Rightarrow \frac{3^2}{(\sqrt{15})^2} + y^2 = 1$$

$$\Rightarrow \frac{9}{15} + y^2 = 1$$

$$\Rightarrow y^2 = 1 - \frac{9}{15} = \frac{6}{15} = \left(\sqrt{\frac{6}{15}}\right)^2 = \frac{(\sqrt{6})^2}{(\sqrt{15})^2}$$

$$\Rightarrow y = \pm \frac{\sqrt{6}}{\sqrt{15}}$$

Now, P is in 2nd Quadrant

$$\Rightarrow x = -\frac{\sqrt{2}}{3}$$

And, Q is in 4th Quadrant.

$$\Rightarrow y = -\frac{\sqrt{6}}{\sqrt{15}}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\left(\frac{\sqrt{7}}{\cancel{3}}\right)}{\left(-\frac{\sqrt{2}}{\cancel{3}}\right)} = -\frac{\sqrt{7}}{\sqrt{2}}$$

$$\cot\varphi = \frac{\cos\varphi}{\sin\varphi} = \frac{\left(\frac{3}{\cancel{\sqrt{15}}}\right)}{\left(-\frac{\sqrt{6}}{\cancel{\sqrt{15}}}\right)} = -\frac{3}{\sqrt{6}}$$

Q.3 Find the inverse function of $f(x) = 5^x$, if exists.

First we check: Does the inverse of $f(x)$ exist! So we check if $f(x)$ is one-one or not!

Let $x_1 \neq x_2 \Rightarrow 5^{x_1} \neq 5^{x_2} \Rightarrow f(x_1) \neq f(x_2)$

Therefore, $f(x)$ is one-one & hence $f(x)$ possesses inverse.

So, let $y = 5^x$. Then we interchange the variables,

ie, $x = 5^y$

Then, taking $\log_5$ on both side we get,

$$\log_5 x = \log_5 (5^y) = y = f^{-1}(x)$$

So, $f^{-1}(x) = \log_5 (x)$.

Note: Domain of $f(x) = \mathbb{R}$, the set of all reals.

Range of $f(x) = (0, \infty)$

Domain of $f^{-1}(x) = (0, \infty)$

Range of $f^{-1}(x) = \mathbb{R}$.

Q.4 If the position function (min) of a car is given by $s(t) = t^3 + 4t^2 + 7$, then find the average velocity between $t = 1$ sec & $t = 3$ sec.

Average Velocity between $t=1$ & $t=3$ seconds is given by the Change of position of the car between $t=1$ sec. & $t=3$ sec

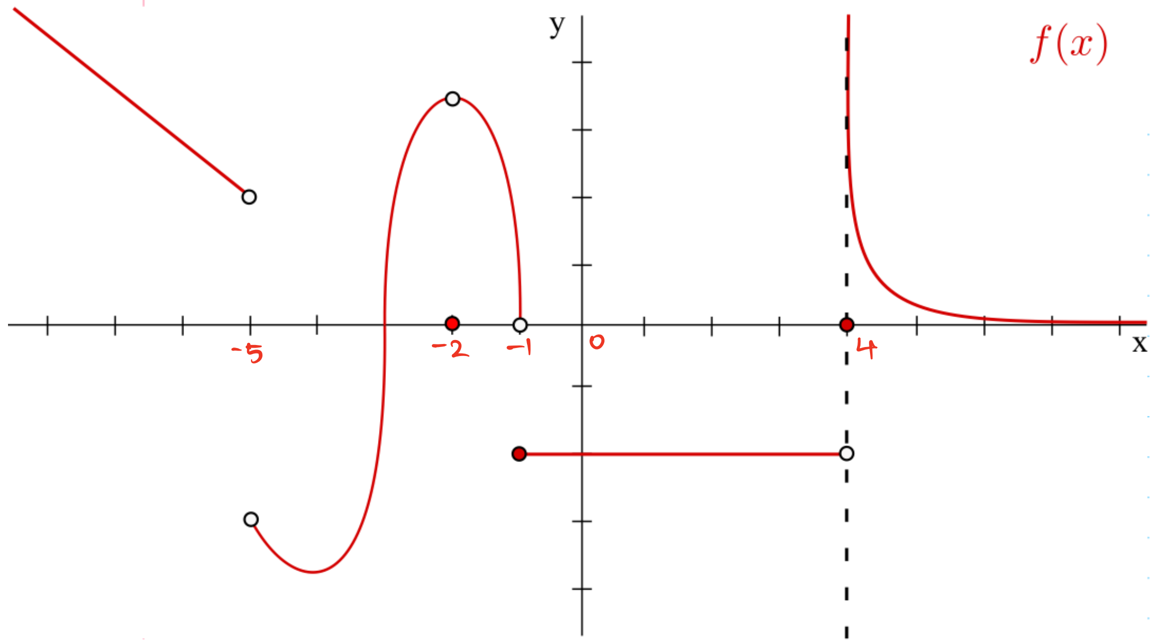
So average velocity $= \dfrac{s(3) - s(1)}{3-1}$ m/sec

$$= \frac{(3^3 + 4.3^2 + 7) - (1^3 + 4.1^2 + 7)}{3-1} \text{ m/sec}$$

$$= \frac{(27+36+7) - (1+4+7)}{2} \text{ m/sec}$$

$$= 29 \text{ m/sec}$$

Q.5.



Analyze the behaviour of $f(x)$ around $x = -5, -2, -1, 0, 4$.

$\lim_{x \to -5^-} f(x) = 2$, $\lim_{x \to -5^+} f(x) = -3$, $f(-5) = DNE$

$\Rightarrow$ $x = -5$ has a jump discontinuity.

$\lim_{x \to -2^-} f(x) = 3.5 = \lim_{x \to -2^+} f(x)$ & $f(-2) = 0$

$\Rightarrow \lim_{x \to -2} f(x) = 3.5$

$\Rightarrow$ $x = -2$ has removable discontinuity.

$\lim_{x \to -1^-} f(x) = 0$, $\lim_{x \to -1^+} f(x) = -2 = f(-1)$

$\Rightarrow$ $x = -1$ has jump discontinuity.

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0) = -2$$

$\Rightarrow$ f is continuous at $x = 0$.

$$\lim_{x\to 4^-} f(x) = -2, \quad \lim_{x\to 4^+} f(x) = +\infty \quad , \quad f(4) = 0$$

$\Rightarrow$ $x = 4$ is an infinite discontinuity.

Q. 6. Evaluate ① $\lim\limits_{x\to \cancel{4}\,3} \sin\left(\log_7 \sqrt{4-x}\right)$

② $\lim\limits_{x\to 4^+} \sin\left(\log_7 \sqrt{4-x}\right)$

③ $\lim\limits_{x\to 4} \left(f(x) + 8g(x) - 4\right)$, where $\lim\limits_{x\to 4} f(x) = 1$,

$\lim\limits_{x\to 8} f(x) = 4$, $\lim\limits_{x\to 4} g(x) = 0$, $\lim\limits_{x\to -4} g(x) = 7$

① $f(x) = \sin\left(\log_7 \sqrt{4-x}\right)$, domain of f is $[0, 4)$

Then $\lim\limits_{x\to 3} f(x) = \lim\limits_{x\to 3} \sin\left(\log_7 \sqrt{4-x}\right)$

$$= \sin\left(\log_7 \sqrt{4 - \lim_{x\to 3} x}\right)$$

$$= \sin\left(\log_7 \sqrt{4-3}\right)$$

$$= \sin\left(\log_7 1\right)$$

$$= \sin(0) = 0.$$

② Since $x \to 4^+$, $f(x)$ is not defined for any value greater than or equal to 4

Hence, $\lim\limits_{x\to 4^+} \sin\left(\log_7 \sqrt{4-x}\right) = \text{DNE}$.

③ $\lim\limits_{x\to 4} \left(f(x) + 8g(x) - 4\right) = \lim\limits_{x\to 4} f(x) + 8 \lim\limits_{x\to 4} g(x) - 4$

$$= 1 + 8(0) - 4 = -3$$

Q.7. Analyze continuity of $h(x) = \frac{x^2+1}{x^2+x-12}$, $x \in [0,5]$.

Domain of $h(x) = \{x \in [0,5] \mid x^2+x-12 \neq 0\}$

$= \{x \in [0,5] \mid (x+4)(x-3) \neq 0\}$

$= \{x \in [0,5] \mid x \neq -4, 3\}$

$= \{x \in [0,5] \mid x \neq 3\}$, as $-4 \notin [0,5]$

So, $x=3$ is a point of discontinuity, &

$\lim_{x \to 3^-} h(x) = \lim_{x \to 3^-} \frac{x^2+1}{x^2+x-12} = -\infty$, as $h(2)<0, \ldots, h(2.99)<0$

(can be checked by calc.)

$\lim_{x \to 3^+} h(x) = \lim_{x \to 3^+} \frac{x^2+1}{x^2+x-12} = +\infty$, as $h(4)>0, \ldots, h(3.01)>0$

(can be checked by calc.)

$\Rightarrow$ $x=3$ has an infinite discontinuity.